problem session 4, part 2

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6 p 134 $\mathbb{Q}[x]/(x^2-2)$ x^2-2 is irreducible in $\mathbb{Q}[x]$ $\mathbb{Q}$ - rationals

therefore $\mathbb{Q}[x]/(x^2-2)$ is a field - Th 5.10

$\mathbb{Q}[x]/(x^2-2)$, as a set, is the set of congruence classes

Every congruence class can be written as $[ax+b]$ with $a, b \in \mathbb{Q}$

because for every $f \in \mathbb{Q}[x]$, by Euclid's Lemma,

$f = q\cdot(x^2-2)+r$ $q \in \mathbb{Q}[x]$ and either $r=0$ or $\deg r < \deg(x^2-2) = 2$

Possible remaiders:

$\{0$, polynomials of degrees 0 and $1\}$

$= \{ax+b \mid a, b \in \mathbb{Q}\}$

Thus for every $f \in \mathbb{Q}[x]$, we have $f \equiv ax+b \pmod{x^2-2}$

$[f] = [ax+b]$ with some $a, b \in \mathbb{Q}$.

Two different polynomials of the form $ax+b$ cannot be congruent modulo x^2-2 because their difference has a degree which is 0 or 1, and as such cannot be divisible by a polynomial of degree 2.

Description of the operations in $\mathbb{Q}[x]/(x^2-2) = \{ [ax+b] \mid a, b \in \mathbb{Q} \}$

Addition

$$[ax+b] + [cx+d] = [\,(a+c)x + (b+d)\,]$$

Multiplication

$$\underline{[ax+b][cx+d]} = [\,(ax+b)(cx+d)\,] = [\,acx^2 + (ad+bc)x + bd\,]$$

$$- ac\,(x^2-2)$$

$$= \underline{[\,(ad+bc)x + (bd+2ac)\,]}$$

If the field $F = \mathbb{Q}[x]/(x^2-2)$ $\quad 0_F = [0] \quad a = b = 0$

$1_F = [1] \quad a = 0,\ b = 1$

$$\underset{a=1\ \ b=0}{[x][x] = [1\cdot x + 0]}\ \underset{c=1\ \ d=0}{[1\cdot x + 0]} = [\,0\cdot x + (0+2)\,] = [2] \quad \Big\} \ [x]^2 = [2] \text{ in } \mathbb{Q}[x]/(x^2-2)$$

Alternatively, one can write $a\sqrt{2}+b$ instead of $[ax+b]$

We denoted $\sqrt{2} = [x]$ $\quad \Big\}$ Could be also $-\sqrt{2} = [x]$

$$\mathbb{Q}[x]/(x^2-2) = \{ a\sqrt{2} + b \mid a, b \in \mathbb{Q} \}$$

11 p134 $\quad R = \mathbb{Q}[x]/(x^2)$ is not a field $\quad$ - Th 5.10 because $x^2 \in \mathbb{Q}[x]$ is not irreducible

Pf . straightforward check using the concreet description.

$\mathbb{Q}[x]/(x^2) = \{ [ax+b] \mid a, b \in \mathbb{Q} \}$

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Multiplication

$$[ax+b][cx+d] = [acx^2 + (ad+bc)x + bd] \quad - acx^2$$

$$= [(ad+bc)x + bd]$$

R is not a field - not even an integral domain

$$[x][x] = [1 \cdot x + 0][1 \cdot x + 0] = [0 \cdot x + 0] = [0]$$

$a=1 \quad b=0 \quad c=1 \quad d=0$

While $[x] \neq [0]$, $\quad [x] \cdot [x] = [0]$ - not an integral domain

Alternative description of $R = \mathbb{Q}[x]/(x^2)$

$R = \{ (a, b) \mid a, b \in \mathbb{Q} \}$ $\quad$ with operations:

$(a, b) + (c, d) = (a+c, b+d)$

$$(a,b)(c,d) = (ad+bc, bd)$$

$$0_R = (0,0)$$

$$1_R = (0,1) \qquad 1_R \neq 0_R$$

R is a commutative ring with identity.

Zero-divisor:

$$(1,0)(1,0) = (0,0)$$